Oxford Cambridge and RSA

Tuesday 20 June 2023 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

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Time allowed: 2 hours

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INFORMATION
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Turn over

Approximating series

Powers of natural numbers

The sum of the first n natural numbers, $1+2+3+...+n$, can be worked out using the formula for the sum of an arithmetic series.

The sum of the squares of the first n natural numbers, $1^2+2^2+3^2+...+n^2$, can be expressed exactly as a formula, $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. There are also exact formulae for the sum of the cubes and for higher powers.

However, the sum of the reciprocals of the first n natural numbers, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$, cannot be expressed exactly in terms of n and only approximate formulae can be found. This particular series is called the harmonic series.

Euler's approximate summation formula

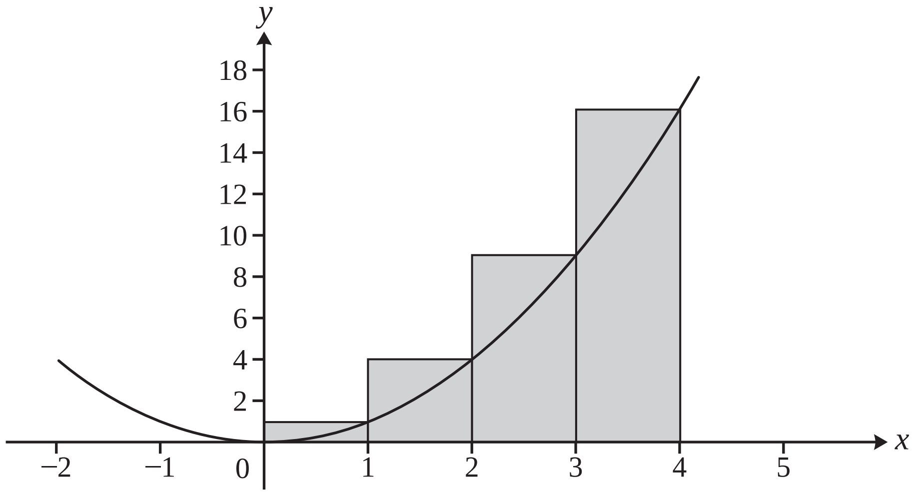
In 1741, the mathematician Leonhard Euler published an approximate formula for summing a series. In modern notation, this can be expressed as follows.

$$\sum_{r=1}^{n} \mathrm{f}(r) \approx \int_1^n \mathrm{f}(x)\,\mathrm{d}x + \frac{\mathrm{f}(n)+\mathrm{f}(1)}{2} + \frac{\mathrm{f}(1)-\mathrm{f}(2)}{12} - \frac{\mathrm{f}(n)-\mathrm{f}(n+1)}{12}$$

Exploring Euler's approximate summation formula for sums of squares

Using Euler's approximate formula for the sum of squares of natural numbers gives the exact sum of the series.

Euler's formula relates a sum of terms to an integral, and this can be illustrated by considering a suitable graph. For the sum of the squares of natural numbers, this is the graph of $y = x^2$. The diagram shows this curve, with four shaded rectangles of areas 1^2, 2^2, 3^2 and 4^2.



Euler's approximate formula for this case is as follows.

$$\sum_{r=1}^{4} r^2 \approx \int_1^4 x^2\,\mathrm{d}x + \frac{4^2+1^2}{2} + \frac{1^2-2^2}{12} - \frac{4^2-5^2}{12}$$

The integral gives the area under the curve between $x = 1$ and $x = 4$. It is clear that the integral is smaller than $\sum_{r=1}^{4} r^2$ so something needs to be added to the integral to get the same answer as the series. The rectangle for 1^2 needs to be added on and so do the parts of the other three rectangles that are above the curve.

Approximating the curve by a series of straight lines gives three triangles to be added on. These have areas $\frac{2^2-1^2}{2}$, $\frac{3^2-2^2}{2}$ and $\frac{4^2-3^2}{2}$.

This gives an approximation for the series of $\int_1^4 x^2\,\mathrm{d}x + 1^2 + \frac{2^2-1^2}{2} + \frac{3^2-2^2}{2} + \frac{4^2-3^2}{2}$ which simplifies to $\int_1^4 x^2\,\mathrm{d}x + \frac{4^2+1^2}{2}$. The final two terms in Euler's approximate formula are to correct for the curve not being a series of straight lines.

Applying Euler's approximate summation formula to the harmonic series

Using Euler's approximate summation for the harmonic series gives

$$\sum_{r=1}^{n} \frac{1}{r} \approx \int_1^n \frac{1}{x}\,\mathrm{d}x + \frac{1}{2}\left(\frac{1}{n}+1\right) + \frac{1}{12}\left(1-\frac{1}{2}\right) - \frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

This simplifies to $\sum_{r=1}^{n} \frac{1}{r} \approx \ln n + \frac{13}{24} + \frac{6n+5}{12n(n+1)}$.